N-Fractional Calculus of Some Multi-Powers Functions(Study on Geometric Univalent Function Theory)

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N- Fractional Calculus of Some Multi- Powers Functions

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Abstract

In a previous article of the author, N- fractional calculus

$$\left(((z-b)^{\beta}-c)^{\alpha}\right)_{\gamma}, \quad ((z-b)^{\beta}-c \neq 0)$$

are discussed. In this article that of more extended forms

$$\left((((z-b)^{\beta}-c)^{\alpha}-d)^{\delta}\right)_{\gamma}, \quad (((z-b)^{\beta}-c)^{\alpha}-d \neq 0)$$

are discussed. Moreover their special cases

$$\left((((z-b)^{\beta}-c)^{\alpha}-d)^{\delta}\right)_{n}, \quad (n \in Z_0^{+}, ((z-b)^{\beta}-c)^{\alpha}-d \neq 0)$$

are presented .

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\,\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\,\mathrm{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D (z \in D)$,

$$f_{\nu} = (f)_{\nu} = {}_{C}(f)_{\nu} = \frac{\Gamma(\nu+1)}{2\pi i}\int_{C}\frac{f(\zeta)}{(\zeta-z)^{\nu+1}}d\zeta \qquad (\nu \notin Z^{-}), \tag{1}$$

$$(f)_{-m} = \lim_{\nu \to -m}(f)_{\nu} \qquad (m \in Z^{+}), \tag{2}$$

where $-\pi \leq \arg(\zeta - z) \leq \pi$ for C_- , $0 \leq \arg(\zeta - z) \leq 2\pi$ for C_+ ,

$\zeta \neq z$, $z \in C$, $\nu \in R$, Γ ; Gamma function,

then $(f)_{\nu}$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_{\nu}| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. *Let fractional calculus operator (Nishimoto's Operator) N^ν be*

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\zeta}{(\zeta - z)^{\nu+1}} \right) \qquad (\nu \notin Z^-), \quad [\,Refer\ to\,(1)\,] \tag{3}$$

with

$$N^{-m} = \lim_{\nu \to -m} N^\nu \qquad (m \in Z^+), \tag{4}$$

and define the binary operation $\circ$ *as*

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta (N^\alpha f) \qquad (\alpha, \beta \in R), \tag{5}$$

then the set

$$\{N^\nu\} = \{N^\nu \mid \nu \in R\} \tag{6}$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{ f;\ 0 \neq |f_\nu| < \infty, \nu \in R \}$, where $f = f(z)$ and $z \in C$.
(vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α.)

Theorem B. " F.O.G. $\{N^\nu\}$ " *is an " Action product group which has continuous index ν " for the set of F . (F.O.G.* ; Fractional calculus operator group) [3]

(III) Lemma. We have [1]

(i) $$((z-c)^b)_\alpha = e^{-i\pi\alpha} \frac{\Gamma(\alpha - b)}{\Gamma(-b)} (z-c)^{b-\alpha} \qquad \left(\left| \frac{\Gamma(\alpha - b)}{\Gamma(-b)} \right| < \infty \right), \tag{7}$$

(ii) $$(\log(z-c))_\alpha = -e^{-i\pi\alpha} \Gamma(\alpha)(z-c)^{-\alpha} \qquad (|\Gamma(\alpha)| < \infty), \tag{8}$$

(iii) $$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \qquad (|\Gamma(\alpha)| < \infty), \tag{9}$$

where $z - c \neq 0$ for (i) and $z - c \neq 0, 1$ for (ii), (iii) ,

(iv) $$(u \cdot v)_\alpha := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)} u_{\alpha-k} v_k \qquad \left(\begin{matrix} u = u(z), \\ v = v(z) \end{matrix} \right). \tag{10}$$

§ 1. Preliminary

The Teorem below is reported by the author already (cf. J. Frac. Calc. Vol. 29, May (2006),pp.35 - 44.) . [12]

Theorem D. *We have*

(i) $$\left(((z-b)^\beta - c)^\alpha \right)_\gamma = e^{-i\pi\gamma} (z-b)^{\alpha\beta - \gamma}$$

$$\times \sum_{k=0}^{\infty} \frac{[-\alpha]_k \Gamma(\beta k - \alpha\beta + \gamma)}{k! \, \Gamma(\beta k - \alpha\beta)} \left(\frac{c}{(z-b)^{\beta}} \right)^k \tag{1}$$

$$\left(\left| \frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)} \right| < \infty \right)$$

and

(ii) $$\left(((z-b)^{\beta} - c)^{\alpha} \right)_n = (-1)^n (z-b)^{\alpha\beta - n}$$

$$\times \sum_{k=0}^{\infty} \frac{[-\alpha]_k [\beta k - \alpha\beta]_n}{k!} \left(\frac{c}{(z-b)^{\beta}} \right)^k \qquad (n \in \mathbf{Z}_0^+) \tag{2}$$

where

$$\left| \frac{c}{(z-b)^{\beta}} \right| < 1 ,$$

and

$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda)$ *with* $[\lambda]_0 = 1$,

(Notation of Pochhammer).

§ 2. N- Fractional Calculus of Functions $\left(((z-b)^{\beta} - c)^{\alpha} - d \right)^{\delta}$

Theorem 1. *We have*

(i) $$\left((((z-b)^{\beta} - c)^{\alpha} - d)^{\delta} \right)_{\gamma} = e^{-i\pi\gamma} (z-b)^{\alpha\beta\delta - \gamma}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-\delta]_m [-\alpha(\delta-m)]_k \Gamma(\beta k - \alpha\beta(\delta-m) + \gamma)}{m! \cdot k! \, \Gamma(\beta k - \alpha\beta(\delta-m))} \left(\frac{c}{(z-b)^{\beta}} \right)^k \left(\frac{d}{(z-b)^{\alpha\beta}} \right)^m \tag{1}$$

$$\left(\left| \frac{\Gamma(\beta k - \alpha\beta(\delta-m) + \gamma)}{\Gamma(\beta k - \alpha\beta(\delta-m))} \right| < \infty \right)$$

and

(ii) $$\left(((z-b)^{\beta} - c)^{\alpha} - d)^{\delta} \right)_n = (-1)^n (z-b)^{\alpha\beta\delta - n}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-\delta]_m [-\alpha(\delta-m)]_k [\beta k - \alpha\beta(\delta-m)]_n}{m! \cdot k!} \left(\frac{c}{(z-b)^{\beta}} \right)^k \left(\frac{d}{(z-b)^{\alpha\beta}} \right)^m , \tag{2}$$

where

$$((z-b)^{\beta}-c)^{\alpha}-d \neq 0, \quad \left|\frac{c}{(z-b)^{\beta}}\right|<1, \quad \left|\frac{d}{(z-b)^{\alpha\beta}}\right|<1, \quad n \in Z_0^+.$$

Proof of (i). We have

$$(((z-b)^{\beta}-c)^{\alpha}-d)^{\delta} = X^{\delta}(1-\tfrac{d}{X})^{\delta} \qquad (X=((z-b)^{\beta}-c)^{\alpha}) \tag{3}$$

$$= X^{\delta}\sum_{m=0}^{\infty}\frac{[-\delta]_m}{m!}\left(\frac{d}{X}\right)^m \tag{4}$$

$$= \sum_{m=0}^{\infty}\frac{[-\delta]_m(d^m)}{m!}X^{\delta-m} \tag{5}$$

$$= \sum_{m=0}^{\infty}\frac{[-\delta]_m(d^m)}{m!}\left((z-b)^{\beta}-c\right)^{\alpha(\delta-m)}, \tag{6}$$

hence, operating N- fractional calculus operator N^{γ} to the both sides of (6), we obtain

$$\left((((z-b)^{\beta}-c)^{\alpha}-d)^{\delta}\right)_{\gamma}$$

$$= \sum_{m=0}^{\infty}\frac{[-\delta]_m(d^m)}{m!}\left(((z-b)^{\beta}-c)^{\alpha(\delta-m)}\right)_{\gamma} \tag{7}$$

$$= \sum_{m=0}^{\infty}\frac{[-\delta]_m(d^m)}{m!}\Big[e^{-i\pi\gamma}(z-b)^{\alpha\beta(\delta-m)-\gamma}$$

$$\times\sum_{k=0}^{\infty}\frac{[-\alpha(\delta-m)]_k\,\Gamma(\beta k-\alpha\beta(\delta-m)+\gamma)}{k!\,\Gamma(\beta k-\alpha\beta(\delta-m))}\left(\frac{c}{(z-b)^{\beta}}\right)^k, \tag{8}$$

applying Theorem D. (i), under the conditions stated before.

We have then (1) from (8) clearly.

Proof of (i i). Set $\gamma=n$ in (1).

Note 1. We use the notations $\sum_{m,k=0}^{\infty}\cdots = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\cdots$, for our convenience.

Note.2. When $d=0$ and $\delta=1$, Theorem 1 is reduced to Theorem D., clearly.

Corollary 1. *We have*

(i) $$\left((((z^{\beta}-c)^{\alpha}-d)^{\delta}\right)_{\gamma}=e^{-i\pi\gamma}z^{\alpha\beta\delta-\gamma}$$

$$\times\sum_{m,k=0}^{\infty}\frac{[-\delta]_m[-\alpha(\delta-m)]_k\Gamma(\beta k-\alpha\beta(\delta-m)+\gamma)}{m!\cdot k!\,\Gamma(\beta k-\alpha\beta(\delta-m))}\left(\frac{c}{z^{\beta}}\right)^k\left(\frac{d}{z^{\alpha\beta}}\right)^m \tag{9}$$

$$\left(\left|\frac{\Gamma(\beta k-\alpha\beta(\delta-m)+\gamma)}{\Gamma(\beta k-\alpha\beta(\delta-m))}\right|<\infty\right)$$

and

(ii) $$\left(((z^{\beta}-c)^{\alpha}-d)^{\delta}\right)_n=(-1)^n z^{\alpha\beta\delta-n}$$

$$\times\sum_{m,k=0}^{\infty}\frac{[-\delta]_m[-\alpha(\delta-m)]_k[\beta k-\alpha\beta(\delta-m)]_n}{m!\cdot k!}\left(\frac{c}{z^{\beta}}\right)^k\left(\frac{d}{z^{\alpha\beta}}\right)^m, \tag{10}$$

where

$$(z^{\beta}-c)^{\alpha}-d\neq 0,\quad \left|\frac{c}{z^{\beta}}\right|<1,\quad \left|\frac{d}{z^{\alpha\beta}}\right|<1,\quad n\in Z_0^{+}.$$

Proof. Set $b=0$ in Theorem 1.

§ 3. Some Special Cases

[I] When $\beta=\alpha=1$, we obtain

$$\left((((z-b-c-d)^{\delta}\right)_{\gamma}=e^{-i\pi\gamma}(z-b)^{\delta-\gamma}$$

$$\times\sum_{m,k=0}^{\infty}\frac{[-\delta]_m[m-\delta]_k\Gamma(k+m-\delta+\gamma)}{m!\cdot k!\,\Gamma(k+m-\delta)}\left(\frac{c}{z-b}\right)^k\left(\frac{d}{z-b}\right)^m \tag{1}$$

$$\left(\left|\frac{c}{z-b}\right|,\ \left|\frac{d}{z-b}\right|<1,\quad \left|\frac{\Gamma(k+m-\delta+\gamma)}{\Gamma(k+m-\delta)}\right|<\infty\right)$$

from Theorem 1. (i).

Now we have the identities

$$\Gamma(m-\delta)=\Gamma(-\delta)[-\delta]_m\quad, \tag{2}$$

$$\Gamma(k+m-\delta)=\Gamma(m-\delta)[m-\delta]_k\quad, \tag{3}$$

$$\Gamma(k+m-\delta+\gamma)=\Gamma(m-\delta+\gamma)[m-\delta+\gamma]_k\quad, \tag{4}$$

then applying (2) ~ (4) into the RHS of (1), we obtain

$$\text{RHS of (1)} = e^{-i\pi\gamma}(z-b)^{\delta-\gamma}$$

$$\times \sum_{m=0}^{\infty} \frac{\Gamma(m-\delta+\gamma)}{m!\,\Gamma(-\delta)}\left(\frac{d}{z-b}\right)^m \sum_{k=0}^{\infty} \frac{[m-\delta+\gamma]_k}{k!}\left(\frac{c}{z-b}\right)^k \tag{5}$$

$$= e^{-i\pi\gamma}(z-b)^{\delta-\gamma} \sum_{m=0}^{\infty} \frac{\Gamma(m-\delta+\gamma)}{m!\,\Gamma(-\delta)}\left(\frac{d}{z-b}\right)^m \left(1-\frac{c}{z-b}\right)^{\delta-\gamma-m} \tag{6}$$

$$= e^{-i\pi\gamma}(z-b-c)^{\delta-\gamma} \frac{\Gamma(-\delta+\gamma)}{\Gamma(-\delta)} \sum_{m=0}^{\infty} \frac{[-\delta+\gamma]_m}{m!}\left(\frac{d}{z-b-c}\right)^m \tag{7}$$

$$= e^{-i\pi\gamma}\frac{\Gamma(-\delta+\gamma)}{\Gamma(-\delta)}(z-b-c-d)^{\delta-\gamma}\,, \qquad \left(\left|\frac{\Gamma(-\delta+\gamma)}{\Gamma(-\delta)}\right| < \infty\right) \tag{8}$$

using the relationship

$$\sum_{k=0}^{\infty} \frac{[\lambda]_k}{k!} z^k = (1-z)^{-\lambda}\,. \tag{9}$$

The result (8) is same as the one obtained by Lemma (i).

[II] When $d=0$ and $\delta=1$, we obtain

$$\left(((z-b)^{\beta}-c)^{\alpha}\right)_{\gamma} = e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-1]_m[-\alpha(1-m)]_k\,\Gamma(\beta k-\alpha\beta(1-m)+\gamma)}{m!\cdot k!\,\Gamma(\beta k-\alpha\beta(1-m))}\left(\frac{c}{(z-b)^{\beta}}\right)^k\left(\frac{0}{(z-b)^{\alpha\beta}}\right)^m \tag{10}$$

$$= e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma} \sum_{k=0}^{\infty} \frac{[-\alpha]_k\,\Gamma(\beta k-\alpha\beta+\gamma)}{k!\,\Gamma(\beta k-\alpha\beta)}\left(\frac{c}{(z-b)^{\beta}}\right)^k \tag{11}$$

$$\left(\left|\frac{c}{(z-b)^{\beta}}\right| < 1,\ \left|\frac{\Gamma(\beta k-\alpha\beta+\gamma)}{\Gamma(\beta k-\alpha\beta)}\right| < \infty\right)$$

from Theorem 1. (i).

That is, in this case we have Theorem D. (i) it self, clearly.

[III] When $n=0$, we have

$$\left(((z-b)^{\beta}-c)^{\alpha}-d)^{\delta}\right)_0 = (z-b)^{\alpha\beta\delta}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-\delta]_m[-\alpha(\delta-m)]_k}{m!\cdot k!}\left(\frac{c}{(z-b)^{\beta}}\right)^k\left(\frac{d}{(z-b)^{\alpha\beta}}\right)^m, \tag{12}$$

$$((z-b)^{\beta}-c)^{\alpha}-d\neq 0\,,\quad \left|\frac{c}{(z-b)^{\beta}}\right|<1,\quad \left|\frac{d}{(z-b)^{\alpha\beta}}\right|<1,\,.$$

from Theorem 1. (i i).

Indeed we obtain

RHS of (12) $=(z-b)^{\alpha\beta\delta}$

$$\times\sum_{m=0}^{\infty}\frac{[-\delta]_m}{m!}\left(\frac{d}{(z-b)^{\alpha\beta}}\right)^m\sum_{k=0}^{\infty}\frac{[-\alpha(\delta-m)]_k}{k!}\left(\frac{c}{(z-b)^{\beta}}\right)^k \qquad (13)$$

$$=(z-b)^{\alpha\beta\delta}\sum_{m=0}^{\infty}\frac{[-\delta]_m}{m!}\left(\frac{d}{(z-b)^{\alpha\beta}}\right)^m\left(1-\frac{c}{(z-b)^{\beta}}\right)^{\alpha\delta-\alpha m} \qquad (14)$$

$$=((z-b)^{\beta}-c)^{\alpha\delta}\sum_{m=0}^{\infty}\frac{[-\delta]_m}{m!}\left(\frac{d}{((z-b)^{\beta}-c)^{\alpha}}\right)^m \qquad (15)$$

$$=((z-b)^{\beta}-c)^{\alpha\delta}\left(1-\frac{d}{((z-b)^{\beta}-c)^{\alpha}}\right)^{\delta} \qquad (16)$$

$$=(((z-b)^{\beta}-c)^{\alpha}-d)^{\delta}\quad , \qquad (17)$$

clearly.

[IV] When $n=1$, we have

$$\left(((z-b)^{\beta}-c)^{\alpha}-d)^{\delta}\right)_1=-(z-b)^{\alpha\beta\delta-1}$$

$$\times\sum_{m,k=0}^{\infty}\frac{[-\delta]_m[-\alpha(\delta-m)]_k\cdot(\beta k-\alpha\beta\,\delta+\alpha\,\beta m)}{m!\cdot k!}\left(\frac{c}{(z-b)^{\beta}}\right)^k\left(\frac{d}{(z-b)^{\alpha\beta}}\right)^m\ , \qquad (18)$$

from Theorem 1. (i i).

Then letting

$$R:=\left(\frac{u}{(z-b)^{\beta}}\right)^{\alpha}\qquad (u=(z-b)^{\beta}-c) \qquad (19)$$

and

$$S:=\left(1-\frac{d}{u^{\alpha}}\right)^{\delta} \qquad (20)$$

we have

$$\sum_{m,k=0}^{\infty} \frac{[-\delta]_m[-\alpha(\delta-m)]_k \cdot \beta k}{m!\cdot k!}\left(\frac{c}{(z-b)^{\beta}}\right)^{k}\left(\frac{d}{(z-b)^{\alpha\beta}}\right)^{m}$$

$$= \frac{-c\alpha\beta\delta}{u} R^{\delta} S - \frac{cd\alpha\beta\delta}{u(u^{\alpha}-d)} R^{\delta} S \quad , \tag{21}$$

$$-\alpha\beta\delta \sum_{m,k=0}^{\infty} \frac{[-\delta]_m[-\alpha(\delta-m)]_k}{m!\cdot k!}\left(\frac{c}{(z-b)^{\beta}}\right)^{k}\left(\frac{d}{(z-b)^{\alpha\beta}}\right)^{m}$$

$$= -\alpha\beta\delta R^{\delta} S \quad , \tag{22}$$

and

$$\alpha\beta \sum_{m,k=0}^{\infty} \frac{[-\delta]_m \cdot m[-\alpha(\delta-m)]_k}{m!\cdot k!}\left(\frac{c}{(z-b)^{\beta}}\right)^{k}\left(\frac{d}{(z-b)^{\alpha\beta}}\right)^{m}$$

$$= -\frac{d\alpha\beta\delta}{(u^{\alpha}-d)} R^{\delta} S \quad , \tag{23}$$

Therefore, applying (23), (22) and (21) into (18), we obtain

$$\left(((z-b)^{\beta}-c)^{\alpha}-d)^{\delta}\right)_1 = (z-b)^{\alpha\beta\delta-1} R^{\delta} S$$

$$\times\left[\frac{c\alpha\beta\delta}{u} + \frac{cd\alpha\beta\delta}{u(u^{\alpha}-d)} + \alpha\beta\delta + \frac{d\alpha\beta\delta}{u^{\alpha}-d}\right] \tag{24}$$

$$= \alpha\beta\delta(z-b)^{\alpha\beta\delta-1}\left(\frac{c}{u}+1\right)\left(\frac{d}{u^{\alpha}-d}+1\right) \tag{25}$$

$$= \alpha\beta\delta(z-b)^{-1}(u^{\alpha}-d)^{\delta}\left(\frac{(z-b)^{\beta}}{u}\right)\left(\frac{u^{\alpha}}{u^{\alpha}-d}\right) \tag{26}$$

$$= \alpha\beta\delta(u^{\alpha}-d)^{\delta-1}(u^{\alpha-1})(z-b)^{\beta-1} \tag{27}$$

$$= \alpha\beta\delta(((z-b)^{\beta}-c)^{\alpha}-d)^{\delta-1}((z-b)^{\beta}-c)^{\alpha-1}(z-b)^{\beta-1} \tag{28}$$

clearly.

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